

Rack shadows and their invariants

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Abstract

A *rack shadow* or *rack set* is a set X with a rack action by a rack R , analogous to a vector space over a field. We use *shadow colorings* of classical link diagrams to define enhanced rack counting invariants and show that the enhanced invariants are stronger than unenhanced counting invariants.

KEYWORDS: quandles, racks, rack shadows, link invariants, enhancements of counting invariants, shadow polynomials

2000 MSC: 57M27, 57M25

1 Introduction

A *rack* is a non-associative algebraic structure whose axioms correspond to the framed Reidemeister moves. *Quandles* are specific types of racks whose axioms correspond to the three unframed Reidemeister moves. Every framed knot or link has a *fundamental rack* described by generators and relations which may be read from any diagram of the knot or link. Similarly, every unframed knot or link has a *fundamental quandle* [6, 3, 7].

The *rack and quandle counting invariants* are integer-valued invariants determined by the number of homomorphisms from the fundamental rack and quandle of a knot or link, which can be pictured as colorings of a knot or link diagram. Invariants of quandle and rack colored knot diagrams define *enhancements* of the counting invariant; enhancements can also make use of extra structure of the coloring quandle or rack. In either case, enhancements specialize to the original counting invariants and in most cases strengthen them. In this paper we will introduce enhancements of the rack counting invariants using sets with right actions by a rack we call *rack shadows* (also known as *rack sets*).

The paper is organized as follows. In section 2 we recall the basics of racks, quandles, and their counting invariants. In section 3 we recall rack shadows and shadow colorings of link diagrams. In section 4 we enhance the shadow counting invariants with shadow polynomials and show that the enhanced invariants contain more information (and are thus stronger) than the unenhanced counting invariants. In section 5 we collect questions for future research.

2 Racks and Quandles

We begin with a definition from [3].

Definition 1 A rack is a set R with two binary operations, \triangleright and \triangleright^{-1} , that satisfy for all $x, y, z \in R$:

- (i) $(x \triangleright y) \triangleright^{-1} y = x$ and $(x \triangleright^{-1} y) \triangleright y = x$, and

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$$(ii) (x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z).$$

Racks can be understood as sets with binary operations in which right “multiplication” by every element is an automorphism. It is a standard exercise to check that in any rack we also have

$$(i) (x \triangleright^{-1} y) \triangleright^{-1} z = (x \triangleright^{-1} z) \triangleright^{-1} (y \triangleright^{-1} z),$$

$$(ii) (x \triangleright y) \triangleright^{-1} z = (x \triangleright^{-1} z) \triangleright (y \triangleright^{-1} z), \text{ and}$$

$$(iii) (x \triangleright^{-1} y) \triangleright z = (x \triangleright z) \triangleright^{-1} (y \triangleright z).$$

Definition 2 A rack in which every $x \in R$ satisfies $x \triangleright x = x$ is a *quandle*.

Example 1 Standard examples of rack structures include:

- (t, s) -racks: Modules over the ring $\tau = \mathbb{Z}[t^{\pm 1}, s]/(s(t + s - 1))$ with rack operation given by

$$x \triangleright y = tx + sy.$$

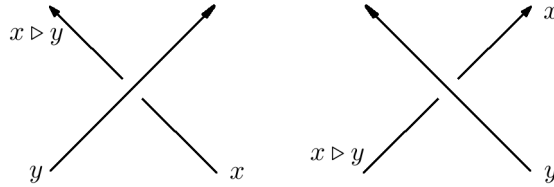
If we set $s = 1 - t$, the result is a quandle known as an *Alexander quandle*.

- *Coxeter racks*: Let \mathbb{F} be a field, V be an \mathbb{F} -vector space and $(,) : V \times V \rightarrow \mathbb{F}$ a symmetric bilinear form. Then the subset $C \subset V$ of nondegenerate vectors is a rack under

$$x \triangleright y = \alpha \left(x - 2 \frac{(x, y)}{(y, y)} y \right)$$

where $0 \neq \alpha \in \mathbb{F}$. If $\alpha = -1$ then C is a quandle.

- *Constant action racks*: For any finite set $R = \{r_1, \dots, r_n\}$ and any permutation $\sigma \in S_n$ setting $r_i \triangleright r_j = r_{\sigma(i)}$ defines a rack operation. If $\sigma = \text{Id}$ we have a quandle called the *trivial* quandle, T_n . T_n is the only quandle in which the \triangleright operation is associative.
- *The Fundamental Rack of a framed link* Let L be an oriented framed link diagram. The fundamental rack of L is the set of equivalence classes of rack words in a set of generators corresponding to arcs in L under the equivalence relation determined by the rack axioms together with the *crossing relations* from the diagram.



As with other universal algebraic objects, we specify a fundamental rack with a presentation $\langle x_1, \dots, x_n \mid r_1, \dots, r_k \rangle$ where x_i are generators and r_i are relations, with the rack axiom relations understood.

To every finite rack $R = \{r_1, \dots, r_n\}$ we associate an $n \times n$ matrix M_R encoding the operation table of the rack, called the *rack matrix*. Specifically, the entry in row i column j of M_R is k where $r_k = r_i \triangleright r_j$.

Example 2 Let $R = \mathbb{Z}_4 = \{1, 2, 3, 4\}$ and set $t = 1$, $s = 2$. Then $2(1 + 2 - 1) = 4 = 0$ and we have a rack operation $x \triangleright y = x + 2y$. The rack matrix is given by

$$M_R = \begin{bmatrix} 3 & 1 & 3 & 1 \\ 4 & 2 & 4 & 2 \\ 1 & 3 & 1 & 3 \\ 2 & 4 & 2 & 4 \end{bmatrix}.$$

It follows easily from the rack axioms that every column in a rack matrix must be a permutation of $\{1, 2, \dots, n\}$. It is also true (see [10], Corollary 2), though perhaps less obvious, that the diagonal of a rack matrix must be a permutation. The *rack rank* of R is the exponent of the diagonal permutation regarded as an element of the symmetric group S_n .

Example 3 The rack in example 2 has rack rank 2, since the diagonal permutation is the transposition (13).

Example 4 Every quandle has diagonal permutation Id_{S_n} and thus rack rank 1. Indeed, a finite rack is a quandle if and only if it has rack rank 1.

Definition 3 Let R and R' be racks. A *rack homomorphism* is a function $f : R \rightarrow R'$ satisfying $f(x \triangleright y) = f(x) \triangleright f(y)$ for all $x, y \in R$. A bijective rack homomorphism is a *rack isomorphism*.

In the case that $R = FR(L)$ is the fundamental rack of an oriented framed link L , a unique homomorphism $f : R \rightarrow R'$ may be specified by assigning an element of R' to each arc of L such that the *crossing condition*

$$\begin{array}{c} \xrightarrow{x \triangleright y} \quad \uparrow \quad \xrightarrow{x} \\ y \end{array}$$

is satisfied at every crossing. Such an assignment of elements of R' to arcs in L is known as a *rack coloring* of L by R' . The relationship between the rack axioms and the framed Reidemeister moves guarantees that rack colorings are preserved by framed Reidemeister moves, in the sense that a rack coloring of a diagram before a move corresponds to a unique rack coloring of the diagram after the move.

Thus, if T is a finite rack and K is a framed oriented knot, the number of rack colorings $|\text{Hom}(FR(K), T)|$ of K by T is an invariant of framed isotopy. In [10] it was shown that for finite racks T , the numbers of colorings of two framings of K are equal if the writhes of the diagrams differ by a multiple of the rack rank N of T . More generally, if $L = L_1 \cup \dots \cup L_k$ is a link with k ordered components, then $w(L_i) \equiv w(L'_i) \pmod{N}$ for $i = 1, \dots, k$ and L ambient isotopic to L' implies that the rack colorings of L and L' by T are in one-to-one correspondence. For a framing vector $\mathbf{w} = (w(L_1), \dots, w(L_k))$ let us abbreviate $\prod_{i=1}^k q_i^{w_i}$ as $q^{\mathbf{w}}$. Then we thus have:

Definition 4 Let $L = L_1 \cup \dots \cup L_k$ be an oriented link with k ordered components, T a finite rack with rack rank N and $W = (\mathbb{Z}_N)^k$. The *integral rack counting invariant* is

$$\text{rc}(L, T) = \sum_{\mathbf{w} \in W} |\text{Hom}(FR(L, \mathbf{w}), T)|$$

and the *polynomial rack counting invariant* is

$$\text{prc}(K, T) = \sum_{\mathbf{w} \in W} |\text{Hom}(FR(L, \mathbf{w}), T)| q^{\mathbf{w}}$$

where (L, \mathbf{w}) is a diagram of L with writhe vector $\mathbf{w} = (w(L_1), \dots, w(L_k))$.

In [10] it is shown that:

Theorem 1 *Let L and L' be oriented links and T a finite rack. If L is ambient isotopic to L' then we have $rc(L, T) = rc(L', T)$ and $prc(L, T) = prc(L', T)$.*

Unlike groups, in which the trivial action is concentrated in a single identity element, trivial action is distributed throughout the structure in racks. We quantify this distribution with a *rack polynomial*.

Definition 5 Let R be a rack. The *rack polynomial* of R is the two-variable polynomial

$$rp(R) = \sum_{x \in R} t^{c(x)} s^{r(x)}$$

where $c(x) = |\{y \in R \mid y \triangleright x = y\}|$ and $r(x) = |\{y \in R \mid x \triangleright y = x\}|$.

We can easily compute the rack polynomial of a rack from its rack matrix by counting the number of occurrences of the row number in each row and column.

Example 5 The rack in example 2 has rack polynomial $rp(R) = 2s^2 + 2t^4s^2$.

In [9] it is shown that:

Theorem 2 *If T and T' are isomorphic racks, then $rp(T) = rp(T')$.*

Given a subrack of a rack R , there is a *subrack polynomial* which captures information both about the subrack itself and how it is embedded in R .

Definition 6 Let $S \subset R$ be a subrack of R . The *subrack polynomial* of $S \subset R$ is given by

$$srp_{S \subset R}(s, t) = \sum_{x \in S} t^{c(x)} s^{r(x)}.$$

That is, we simply sum up the contributions to the full rack polynomial from the elements of the subrack.

Example 6 The subrack $S = \{1, 3\} \subset R$ in example 2 has subrack polynomial $srp_{S \subset R}(s, t) = 2s^2$.

Rack polynomials can be used to define an enhancement of the rack counting invariant, as described in [1]. Specifically, for each rack coloring we find the image subrack and use its subrack polynomial as a “signature” of the coloring. The multiset of such polynomials paired with framing information then yields an enhancement of the rack counting invariant; alternatively we can convert to a “polynomial-style” format.

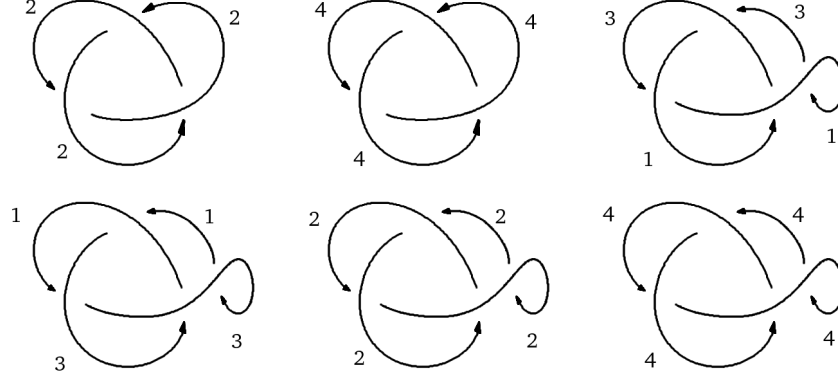
Definition 7 Let $L = L_1 \cup \dots \cup L_k$ be an oriented link with ordered components, $W = (\mathbb{Z}_N)^k$ the space of framing vectors mod N , and T a finite target rack with rack rank N . The *multiset rack polynomial invariant* is the multiset

$$mrp(L, T) = \{(\text{srp}_{\text{Im}(f) \subset T}(s, t), \mathbf{w}) \mid f \in \text{Hom}(FR(L, \mathbf{w}), T), \mathbf{w} \in W\}$$

and the *polynomial rack polynomial invariant* is

$$prp(L, T) = \sum_{\mathbf{w} \in W} \left(\sum_{f \in \text{Hom}(FR(L, \mathbf{w}), T)} z^{\text{srp}_{\text{Im}(f) \subset T}(s, t)} q^{\mathbf{w}} \right).$$

Example 7 The trefoil knot 3_1 below has integral rack counting invariant $\text{rc}(3_1) = 6$, polynomial rack counting invariant $\text{prc}(3_1) = 4 + 2q$ and polynomial rack polynomial invariant $\text{prp}(3_1) = 2z^{2t^2} + 2z^{s^4t^2} + 2qz^{s^4t^2}$ with respect to the rack R from example 2.



3 Rack and Quandle Shadows

In this section we define *rack shadows* (also known as *rack sets*) which can be used to generalize the *shadow colorings* of knot diagrams by quandles described in previous work such as [2, 4, 8].

Definition 8 Let X be a set and R a rack. A *rack action* of R on X is an assignment of bijection $\cdot r : X \rightarrow X$ to each element of R such that for all $x \in X$ and $r_1, r_2 \in R$:

$$(x \cdot r_1) \cdot r_2 = (x \cdot r_2) \cdot (r_1 \triangleright r_2).$$

Definition 9 Let R be a rack. A *rack shadow* or *R -shadow* is a set X with a rack action by a rack R . The *shadow matrix* of $X = \{x_1, \dots, x_m\}$ where $R = \{r_1, \dots, r_n\}$ is the $m \times n$ matrix whose (i, j) entry is k where $x_k = x_i \cdot r_j$. A subset of X closed under the action of R is an *R -subshadow* of X .

Example 8 Let R be any rack and let $X = \{x\}$. Then X is an R -shadow under the shadow operation $x \cdot r = x$ for all $r \in R$, since we have

$$(x \cdot r) \cdot r' = (x \cdot r) = x = (x \cdot r') = (x \cdot r') \cdot (r \triangleright r').$$

We will call this structure the *singleton R -shadow*.

Example 9 Let R be any rack and let $X = R$. Then X is an R -shadow under the shadow operation $x \cdot r = x \triangleright r$ for all $r \in R$, since we have

$$(x \cdot r) \cdot r' = (x \triangleright r) \triangleright r' = (x \triangleright r') \triangleright (r \triangleright r') = (x \cdot r') \cdot (r \triangleright r').$$

We will call this structure the *rack R -shadow*. Note that the shadow matrix of the rack R -shadow is the same as the rack matrix of R .

Example 10 Let R be any rack and $X = \{x_1, \dots, x_n\}$ a set of cardinality n . Then for any permutation $\sigma \in S_n$ we have an R -shadow structure given by $x_i \cdot r = x_{\sigma(i)}$:

$$(x \cdot r) \cdot r' = x_{\sigma(i)} \cdot r' = x_{\sigma^2(i)} = x_{\sigma(i)} \cdot (r \triangleright r') = (x_i \cdot r') \cdot (r \triangleright r').$$

We will call this shadow structure a *constant action shadow*.

Example 11 For a less trivial example of an R -shadow structure, let R be the (t, s) -rack from example 2, i.e. $R = \mathbb{Z}_4$ with $t = 1$, $s = 2$ and $x \triangleright y = tx + sy = x + 2y$. Then the three element set $X = \{1, 2, 3\}$ is an R -shadow with operation matrix

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 3 & 2 & 3 \\ 3 & 1 & 3 & 1 \end{bmatrix}.$$

Definition 10 Let R be a rack. Two R -shadows X, Y are *isomorphic* if there is a bijection $\phi : X \rightarrow Y$ such that $\phi(x \cdot r) = \phi(x) \cdot r$ for all $r \in R$.

Proposition 3 Let R be a rack and let X and Y be n -element constant action R -shadows with permutations $\sigma, \tau \in S_n$ respectively. Then X is isomorphic to Y if and only if σ is conjugate to τ in S_n .

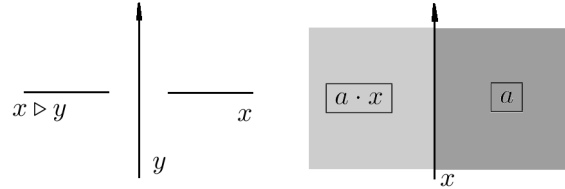
Proof. Suppose $\phi : X \rightarrow Y$ is an R -shadow isomorphism. Then $\phi(x_i) = y_j$ induces a permutation $\Phi \in S_n$ defined by $\Phi(i) = j$, so that $\phi(x_i) = y_{\Phi(i)}$ and we have

$$\phi(x_i \cdot r) = \phi(x_{\sigma(i)}) = y_{\Phi(\sigma(i))} \quad \text{and} \quad \phi(x_i) \cdot r = y_{\Phi(i)} \cdot r = y_{\tau(\Phi(i))}$$

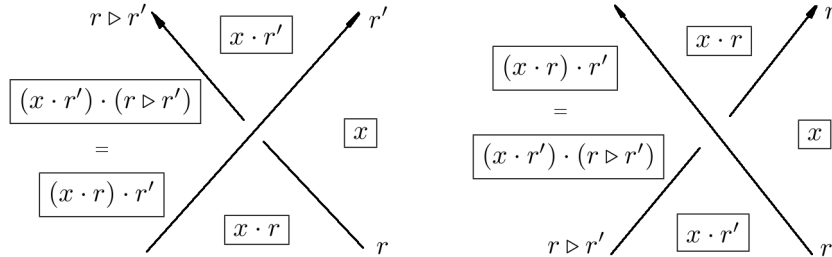
and hence $\Phi \circ \sigma \circ \Phi^{-1} = \tau$.

Conversely, τ conjugate to σ in S_n by Φ in S_n implies that Φ induces an isomorphism $\phi : X \rightarrow Y$ by $\phi(x_i) = y_{\Phi(i)}$. \square

Definition 11 Let L be a framed oriented link diagram and X an R -shadow. A *shadow coloring* of L by R is an assignment of elements of R to arcs in L and elements of X to the regions between the arcs such that at every crossing and along every arc we have



Shadow colors will be indicated by boxes. Note that the requirement that R acts on X via a rack action, i.e. the rack shadow axiom, is precisely the condition needed to guarantee that shadow colorings are well-defined at crossings:



Proposition 4 Let L be a link diagram, R a rack and X an R -shadow. Then for each rack coloring of L by R and each element of X there is exactly one shadow coloring of L .

Proof. This is well-known in the case that $X = R$ is the rack R -shadow; see [8] for instance. For the case of a general R -shadow X , consider a rack coloring of L and choose a region of L . Any element of X can be assigned to the chosen region, and any such choice determines a unique shadow coloring by pushing the color across the arcs with \cdot or \cdot^{-1} . \square

Definition 12 Let L be a link diagram, R a rack and X an R -shadow. The *shadow counting invariant* $\text{sc}(L)$ is the number of shadow colorings of L by X .

Corollary 5 The shadow counting invariant of a link L by an R -shadow X is given by

$$\text{sc}(L) = |X| \text{rc}(L, R).$$

Just as racks have rack polynomials, we can define a polynomial invariant of R -shadows:

Definition 13 The *R -shadow polynomial* $\text{rsp}(X)$ of an R -shadow X is the sum

$$\text{rsp}(X) = \sum_{x \in X} t^{r(x)}$$

where $r(x) = |\{r \in R \mid x \cdot r = x\}|$. If $S \subset X$ is a subshadow then the *subshadow polynomial* of s is

$$\text{ssp}_{S \subset X}(t) = \sum_{x \in S} t^{r(x)}$$

Proposition 6 If two R -shadows X and Y are isomorphic, we have $\text{rsp}(X) = \text{rsp}(Y)$.

Proof.

Suppose $\phi : X \rightarrow Y$ is an R -shadow isomorphism. Then $r(\phi(x)) = r(x)$ and the contribution to $\text{rsp}(X)$ from $x \in X$ is equal to the contribution to $\text{rsp}(Y)$ from $y = \phi(x) \in Y$. Since every $y \in Y$ is equal to $\phi(x)$ for some $x \in X$ and conversely, it follows that the sums are equal. \square

4 Shadow Enhanced Counting Invariants

Unfortunately, corollary 5 implies that the unenhanced shadow counting invariant does not contain any more information than the ordinary rack counting invariant. If we want to exploit rack shadows, then, we must look to enhancements.

For the rack R -shadow $X = R$ in the case that R is quandle, this is done in [2, 8] with cocycles in the third cohomology of the target quandle $H_Q^3(R)$. These invariants have a natural interpretation in terms of quandle colorings of knotted surfaces in \mathbb{R}^4 . In this section we will take a different approach, using R -shadow polynomials to define an enhancement.

Definition 14 For a shadow coloring f of a link diagram L by an R -shadow X , the closure of the set of shadow colors under the action of the image subrack $\text{Im}(f) \subset R$ of f is a subshadow called the *shadow image* of f , denoted $\text{si}(f)$.

It is easy to see that the shadow image of a shadow colored link diagram is invariant under framed Reidemeister moves, and thus its subshadow polynomial can be used as a signature of the shadow coloring. We then have:

Definition 15 Let $L = L_1 \cup L_2 \cup \cdots \cup L_k$ be an oriented link with k components, R a finite rack with rack rank N , $W = (\mathbb{Z}_N)^k$ and X an R -shadow. The *multiset shadow polynomial invariant* of L with respect to the R -shadow X is the multiset

$$\Phi(L) = \{(\text{ssp}_{\text{si}(f) \subset X}(t), \mathbf{w}) \mid \mathbf{w} \in (\mathbb{Z}_N)^k, f \text{ shadow coloring}\}$$

and the R -shadow polynomial invariant of L with respect to X is

$$\text{sp}(L) = \sum_{\mathbf{w} \in W} \left(\sum_{f \text{ shadow coloring}} z^{\text{ssp}_{\text{si}(f)} \subset X(t)} q^{\mathbf{w}} \right).$$

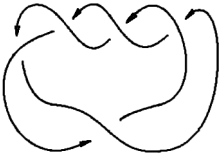
Example 12 If X is the singleton shadow $X = \{x\}$ then $\text{sp}(L) = z^t \text{rp}(L)$ since shadow colorings in this case are simply rack colorings with every region colored \boxed{x} and the subshadow polynomial of every shadow image is equal to t .

Remark 13 The shadow polynomial specializes to the shadow counting invariant $\text{sc}(L, T)$ by setting $t = 0$ (or equivalently $z = 1$). It follows that the shadow polynomial is at least as strong an invariant as the integral rack counting invariant. The following example demonstrates that the shadow polynomial is stronger than the unenhanced shadow counting invariant.

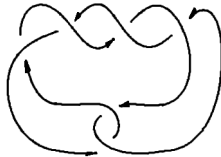
Example 14 The two knots below, 5_1 and 6_1 , both have shadow counting invariant $\text{sc}(5_1, R) = 60 = \text{sc}(6_1, R)$ with respect to the R -shadow $X = \{1, 2\}$ where

$$M_X = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \quad M_R = \begin{bmatrix} 1 & 3 & 5 & 2 & 4 & 3 & 1 & 4 & 2 & 5 \\ 5 & 2 & 4 & 1 & 3 & 5 & 3 & 1 & 4 & 2 \\ 4 & 1 & 3 & 5 & 2 & 2 & 5 & 3 & 1 & 4 \\ 3 & 5 & 2 & 4 & 1 & 4 & 2 & 5 & 3 & 1 \\ 2 & 4 & 1 & 3 & 5 & 1 & 4 & 2 & 5 & 3 \\ 8 & 9 & 10 & 6 & 7 & 6 & 10 & 9 & 8 & 7 \\ 7 & 8 & 9 & 10 & 6 & 8 & 7 & 6 & 10 & 9 \\ 6 & 7 & 8 & 9 & 10 & 10 & 9 & 8 & 7 & 6 \\ 10 & 6 & 7 & 8 & 9 & 7 & 6 & 10 & 9 & 8 \\ 9 & 10 & 6 & 7 & 8 & 9 & 8 & 7 & 6 & 10 \end{bmatrix}.$$

However, the shadow enhanced invariant distinguishes the knots:



$\text{sp}(5_1) = 10z + 10z^t + 40z^t{}^5$



$\text{sp}(6_1) = 50z + 10z^t.$

5 Questions for Further Research

In this section we collect a few questions for further research. Much remains to be done in the study of rack shadows and their invariants.

It is natural to think of a rack action as akin to scalar multiplication. What kinds of further enhancements result when an R -shadow X has extra structure of its own, e.g. when X is a group or another quandle? Work is already underway on quandle/rack homology with coefficients in an R -shadow X , generalizing the 3-cocycle shadow coloring invariants found in [2, 8], for instance.

What is the relationship between the shadow polynomial enhanced invariant determined by an R -shadow X and the rack polynomial enhanced invariant determined by R in [1]? In the special case where $X = R$, the shadow image polynomial is just the subrack polynomial of $\text{Im}(f)$ with $s = 1$.

More generally, every rack can be decomposed as a disjoint union of orbit subracks which act on each other via rack actions (see [11]); what is the relationship between the invariants defined by the orbit racks, the rack shadows, and the overall rack?

Many racks and quandles have additional structure such as abelian groups, modules over various rings, etc. What are some examples of R -shadow structures defined on abelian groups or modules?

`python` code for computing the invariants described in this paper is available from the second author's website at <http://www.esotericka.org>.

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